Announcements

1) Scholarship deadline:
4/10

Definition: (integral in $\mathbb{R}^n$)

Let $E \subseteq \mathbb{R}^n$ be an n-cube.

Let $f: E \to \mathbb{R}$. Let

$$E = \prod_{i=1}^{n} (a_i, b_i) \text{ and}$$

$\{P_i\}_{i=1}^{n}$ partitions of

$\{(a_i, b_i)\}_{i=1}^{n}$, respectively.

If f is bounded on E,

define

$$U(f, \{P_i\}_{i=1}^{n})$$

$$\sum_{j_n=1}^{k_n} \cdots \sum_{j_2=1}^{k_2} \sum_{j_1=1}^{k_1} M_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} \prod_{i=1}^{n} \ell(x_{i_{j_i}}, x_{i_{j_i+1}})$$

where $M_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} = \sup\limits_{x \in \prod_{i=1}^{n} (x_{i_{j_i}}, x_{i_{j_i+1}})} f(x)$

and

$$P_i = \{ \overset{a}{\overset{\|}{x_{i_1}}}, x_{i_2}, \ldots, \overset{b}{\overset{\|}{x_{i_{k_i}}}} \}$$

where k_i = number of subdivisions

Similarly, define

$$L\left(f, \{P_i\}_{i=1}^{n}\right)$$

$$\sum_{j_n=1}^{k_n} \cdots \sum_{j_2=1}^{k_2} \sum_{j_1=1}^{k_1} m_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} \prod_{i=1}^{n} \ell\left(x_{i_{j_i}}, x_{i_{j_i+1}}\right)$$

where $m_{1_{j_1}, 2_{j_2}, \ldots, n_{j_n}} = \inf\limits_{x \in \prod_{i=1}^{n} \left(x_{i_{j_i}}, x_{i_{j_i+1}}\right)} f(x)$

Let $$L(f) = \sup_{\substack{\text{partitions} \\ \{P_i\}_{i=1}^{n}}} L(f, \{P_i\}_{i=1}^{n})$$

and

$$U(f) = \inf_{\substack{\text{partitions} \\ \{P_i\}_{i=1}^{n}}} U(f, \{P_i\}_{i=1}^{n})$$

We say f is

Riemann Integrable

on E if

$$\boxed{U(f) = L(f)}$$

In that case, we set

$\int_E f(x)\,dx$ to be

the common value of $U(f)$ and $L(f)$.

Extension: Let A be a bounded region in $\mathbb{R}^n$ and let $f: A \to \mathbb{R}$. Then A
(f bounded)
is contained in some n-cube E. We define

$$\boxed{\int_A f(x)\,dx = \int_E f(x)\chi_A(x)\,dx}$$

where

$$\chi_A : \mathbb{R}^n \to \mathbb{R},$$

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases},$$

we extend f outside of A by defining it to be identically zero, and provided the latter integral exists!

All familiar results apply

For example, let $A \subseteq \mathbb{R}^n$.
We say A has (Lebesgue) **measure zero** if $\forall\ \varepsilon > 0$, $\exists$ an open cover of A by n-cubes such that the sum of all the volumes of the n-cubes is less than ε.

Then a bounded function f is Riemann-integrable on A if and only if the set of discontinuities of f on A has measure zero. Up to notation, the proof is identical.

Also, if f is integrable over $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ and $A \cap B$ is a set of measure zero, then

$$\int_{A \cup B} f(x)\,dx = \int_A f(x)\,dx + \int_B f(x)\,dx$$

Theorem: (Fubini)

Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$ be n-cubes. Suppose $f: A \times B \to \mathbb{R}$ is bounded and integrable. Define, for a fixed $x \in A$,

$$f_x : B \to \mathbb{R}$$

$$f_x(y) = f(x, y).$$

Then

$$\boxed{\begin{aligned}&\int_{A\times B} f \\ &= \int_A L(f_x, B)\,dx \\ &= \int_A U(f_x, B)\,dx\end{aligned}}$$

Where $L(f_x, B)$ and $U(f_x, B)$ denote the lower and upper sums for f_x over B.

Remarks

1) Integrability: If f is integrable on $A \times B$, this does not necessarily imply f_x is integrable on B for <u>any</u> $x \in A$.

If we do have integrability of f_x off of a set of measure zero contained in A, then Fobini's Theorem becomes

$$\int_{A\times B} f$$

$$= \int_A \left(\int_B f_x(y)\,dy \right) dx$$

If we also define

$f_y : A \to \mathbb{R}$, $f_y(x) = f(x,y)$,

there is an analagous statement of Fubini's theorem for f_y.

If, for example, f is continuous on $A \times B$,

$$\boxed{\begin{aligned} \int_{A\times B} f &= \int_A \left(\int_B f_x \, dy \right) dx \\ &= \int_B \left(\int_A f_y \, dx \right) dy \end{aligned}}$$

2) Non-cubic regions

It is possible, if your region $A \times B$ is "nice" enough, to use Fubini's Theorem on $A \times B$ in the case where A and B are not cubes.

Differential Forms

Definition: (multi-linearity)

A function

$$T: \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}} \longrightarrow \mathbb{R}$$

is said to be multilinear

if $\forall \ 1 \leq i \leq k$,

$$T(x_1, x_2, \ldots, x_{i-1}, x_i + y_i, x_{i+1}, \ldots, x_k)$$

$$= T(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_k)$$
$$+ T(x_1, x_2, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_k)$$

$$T(x_1, x_2, \dots, x_{i-1}, x_i + y_i, x_{i+1}, \dots, x_k)$$

$$= T(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k)$$

$$+ T(x_1, x_2, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_k)$$

and

$$T(x_1, x_2, \dots, x_{i-1}, cx_i, x_{i+1}, \dots, x_k)$$

$$= c\,T(x_1, x_2, \dots, x_k)$$

$\forall\ x_1, x_2, \dots, x_k, y_i \in \mathbb{R}^n$

and $c \in \mathbb{R}$ (T is linear in each variable)

Example 1: (linear)

If $f: \mathbb{R}^n \to \mathbb{R}$ is linear, then f is certainly multilinear!

In fact, f is just multiplication by a $1 \times n$ matrix.

Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$.

$$f\left(\sum_{i=1}^{n} \alpha_i e_i\right) \quad (\alpha_i\text{'s} \in \mathbb{R})$$

$$= \sum_{i=1}^{n} \alpha_i f(e_i), \text{ so}$$

f is determined by its values on the standard basis!

Let $\gamma_i = f(e_i)$, $1 \le i \le n$.

Define $\varphi_i : \mathbb{R}^n \to \mathbb{R}$

$$\varphi_i\left(\sum_{j=1}^{n} \alpha_j e_j\right) = \alpha_i .$$

$(1 \le i \le n)$. Then φ_i is linear, and we can write

$$f\left(\sum_{j=1}^{n} \alpha_j e_j\right)$$

$$= \sum_{i=1}^{n} \gamma_i \varphi_i\left(\sum_{j=1}^{n} \alpha_j e_j\right)$$

So

$\{\varphi_i\}_{i=1}^{n}$ is a basis

for linear maps from

$\mathbb{R}^n$ to $\mathbb{R}$

Linear Independence:

if $\sum_{i=1}^{n} \alpha_i \varphi_i = 0$, then

plug in e_j to get $\alpha_j = 0$

$\forall \; 1 \leq j \leq n$).